AD-A101 636 CALIFORNIA UNIV BERKELEY OPERATIONS RESEARCH CENTER F/G 12/1
SOME APPLICATIONS OF A RESULT CONCERNING VARIABILITY ORDERINGS.(U)
JUN 81 S M ROSS AFOSR-81-0122
UNCLASSIFIED ORC-81-16 NL

1 OF 1
AD A
101636

END
DATE
FILMED
8-81
DTIC

ORC 81-16
JUNE 1981

LEVEL II

12

SOME APPLICATIONS OF A RESULT CONCERNING VARIABILITY ORDERINGS

by

SHELDON M. ROSS

OPERATIONS
RESEARCH
CENTER

UNIVERSITY OF CALIFORNIA · BERKELEY

SOME APPLICATIONS OF A RESULT CONCERNING VARIABILITY ORDERINGS

by

Sheldon M. Ross
Department of Industrial Engineering
and Operations Research
University of California, Berkeley

JUNE 1981 ORC 81-16

This research was supported by the Air Force Office of Scientific Research (AFSC), USAF, under Grant AFOSR-81-0122 with the University of California. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER (14) ORC-81-16	2. GOVT ACCESSION NO. AD-A101 636	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) SOME APPLICATIONS OF A RESULT CONCERNING VARIABILITY ORDERINGS.		5. TYPE OF REPORT & PERIOD COVERED (9) Research Report,
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) (10) Sheldon M. Ross		8. CONTRACT OR GRANT NUMBER(s) (15) AFOSR-81-0122
9. PERFORMING ORGANIZATION NAME AND ADDRESS Operations Research Center University of California Berkeley, California 94720		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS (16) 2304/A5 (17) A5
11. CONTROLLING OFFICE NAME AND ADDRESS United States Air Force Air Force Office of Scientific Research Bolling Air Force Base, D.C. 20332	(11)	12. REPORT DATE June 1981
		13. NUMBER OF PAGES 14 (12) 15
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Variability Ordering
Branching Process
Shock Models

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

(SEE ABSTRACT)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-LF-014-6601

Unclassified 270750

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ABSTRACT

We say that the random variable X is more variable than Y if $E[f(X)] \geq E[f(Y)]$ for all increasing convex functions f . We prove a preservation, under random sized sums, property of this ordering and then apply it to branching processes and shock models.

SOME APPLICATIONS OF A RESULT CONCERNING VARIABILITY ORDERINGS

by

Sheldon M. Ross

1. A VARIABILITY RESULT

If X_1 and X_2 are random variables having respective distributions F_1 and F_2, then we say that $X_1 \underset{v}{\le} X_2$ (read X_1 is less variable than X_2) or equivalently that $F_1 \underset{v}{\le} F_2$ if

$$\int_0^\infty f(x)dF_1(x) \le \int_0^\infty f(x)dF_2(x)$$

for all increasing convex functions f. Some easily derived properties of this ordering are

1. $F_1 \underset{v}{\le} F_2$ if and only if

$$\int_0^\infty \bar{F}_1(x)dx \ge \int_a^\infty \bar{F}_2(x)dx \quad \text{for all } a$$

where $\bar{F}_i = 1 - F_i$.

2. If $F_i \underset{v}{\le} G_i$, $i = 1, 2$ then $F_1 * F_2 \underset{v}{\le} G_1 * G_2$ where $*$ denotes convolution.

We will now present a theorem concerning this ordering and in Sections 2 and 3 apply it to branching processes and shock models.

Theorem 1:

Let $X_1, X_2, \ldots$ be a sequence of nonnegative independent and identically distributed random variables and similarly $Y_1, Y_2, \ldots$. Let N and M be integer valued nonnegative random variables that are independent of the X_i and Y_i sequences. Then

$$X_i \underset{v}{\geq} Y_i \ , \ i \geq 1 \ , \ N \underset{v}{\geq} M \Rightarrow \sum_{i=1}^{N} X_i \geq \sum_{i=1}^{M} Y_i \ .$$

Proof:

We will first show that

$$\sum_{i=1}^{N} X_i \underset{v}{\geq} \sum_{i=1}^{M} X_i \ .$$

Let h denote an increasing convex function. To prove the above we must show that

$$E\left[h\left(\sum_1^N X_i\right)\right] \geq E\left[h\left(\sum_1^M X_i\right)\right] \ . \tag{1}$$

Since $N \underset{v}{\geq} M$, and they are independent of the X_i , the above will follow if we can show that the function $g(n)$, defined by

$$g(n) = E[h(X_1 + \ldots + X_n)]$$

is an increasing convex function of n . As it is clearly increasing since h is and each X_i is nonnegative it remains to show that g is convex, or, equivalently, that

(2) $\qquad g(n+1) - g(n)$ is increasing in n .

To prove this let $S_n = \sum_1^n X_i$, and note that

$$g(n + 1) - g(n) = E[h(S_n + X_{n+1}) - h(S_n)] .$$

Now,

$$E[h(S_n + X_{n+1}) - h(S_n) \mid S_n = t] = E[h(t + X_{n+1}) - h(t)]$$
$$= f(t) \quad \text{(say)}.$$

As h is convex, it follows that $f(t)$ is increasing in t . Also, as S_n increases in n , we see that $E[f(S_n)]$ increases in n . But

$$E[f(S_n)] = g(n + 1) - g(n)$$

and thus (2) and (1) are satisfied.

We have thus proven that

$$\sum_1^N X_i \underset{v}{\geq} \sum_1^M X_i$$

and the proof will be completed by showing that

$$\sum_1^M X_i \underset{v}{\geq} \sum_1^M Y_i$$

or, equivalently, that for increasing, convex h

$$E\left[h\left(\sum_1^M X_i\right)\right] \geq E\left[h\left(\sum_1^M Y_i\right)\right] .$$

But

$$E\left[h\left(\sum_{1}^{M} X_i\right) \mid M = m\right] = E\left[h\left(\sum_{1}^{m} X_i\right)\right] \quad \text{by independence}$$

$$\geq E\left[h\left(\sum_{1}^{m} Y_i\right)\right] \quad \text{since } \sum_{1}^{m} X_i \underset{v}{\geq} \sum_{1}^{m} Y_i$$

$$= E\left[h\left(\sum_{1}^{M} Y_i\right) \mid M = m\right]$$

and the result follows by taking expectations of both sides of the above.||

2. A BRANCHING PROCESS APPLICATION

Consider two Galton Watson branching processes in which individuals at the end of their lifetime give birth to a random number of offspring. Let $X_{jn}^{(i)}$, $j \geq 1$, $n \geq 0$ denote the number of offspring of the j^{th} individual of the n^{th} generation in the i^{th} branching process, $i = 1, 2$. Suppose that the random variables $X_{jn}^{(i)}$, $j \geq 1$, $n \geq 0$ are independent for $i = 1, 2$ and have a distribution not depending on j. In addition, suppose that

$$X_{jn}^{(1)} \underset{v}{\geq} X_{jn}^{(2)} \quad \text{for all } n, j .$$

Let $Z_n^{(i)}$, $i = 1, 2$ denote the size of the n^{th} generation of the i^{th} process.

Proposition 2:

If $Z_0^{(i)} = 1$, $i = 1, 2$, then $Z_n^{(1)} \underset{v}{\geq} Z_n^{(2)}$ for all n.

Proof:

The proof is by induction on n. As it is true for $n = 0$, assume it for n. Now

$$Z_{n+1}^{(1)} = \sum_{j=1}^{Z_n^{(1)}} X_{j,n}^{(1)}$$

$$Z_{n+1}^{(2)} = \sum_{j=1}^{Z_n^{(2)}} X_{j,n}^{(2)}$$

and so the result follows from Theorem 1.||

We now show that if the second (less) variable process has the same mean number of offspring per individual as does the first then it is less likely, at each generation to become extinct.

Corollary 3:

Suppose $E\left[X_{jn}^{(1)}\right] = E\left[X_{jn}^{(2)}\right]$ for all j, n. If $Z_0^{(i)} = 1$, $i = 1, 2$ and $X_{jn}^{(1)} \underset{v}{\geq} X_{jn}^{(2)}$ for all j, n

$$P\left\{Z_n^{(1)} = 0\right\} \geq P\left\{Z_n^{(2)} = 0\right\} \text{ for all } n .$$

Proof:

From Proposition 2 we have that $Z_n^{(1)} \underset{v}{\geq} Z_n^{(2)}$ and thus

$$\sum_{i=2}^{\infty} P\left\{Z_n^{(1)} \geq i\right\} \geq \sum_{i=2}^{\infty} P\left\{Z_n^{(2)} \geq i\right\}$$

or, equivalently, since $E\left[Z_n^{(1)}\right] = \prod_{i=0}^{n-1} E\left[X_{ji}^{(1)}\right] = E\left[Z_n^{(2)}\right] \equiv \mu$

$$\mu - P\left\{Z_n^{(1)} \geq 1\right\} \geq \mu - P\left\{Z_n^{(2)} \geq 1\right\}$$

which proves the result.||

Remarks:

(i) In [2], Freedman and Purves showed that among all branching processes for which $P[X_{jn} = 1] = 0$ and $E[X_{jn}] = M < 2$, all j, n, the one having the least chance of going extinct is the one with $P\{X_{jn} = 0\} = 1 - M/2 = 1 - P\{X_{jn} = 2\}$. This also follows from Corollary 3 upon application of the following lemma (with $\alpha = 0$).

Lemma 4:

Let $P\{X = 1\} = \alpha$, $P\{X = 0\} = (1 - \alpha) - \frac{(M - \alpha)}{2}$, $P\{X = 2\} = \frac{M - \alpha}{2}$ and let Y be a nonnegative, integer valued, and such that $P\{Y = 1\} \leq \alpha$ and $E[Y] = M$. If $\alpha < M < 2 - \alpha$ then $X \underset{v}{\leq} Y$.

Proof:

We must show that

$$\sum_{i=n+1}^{\infty} P\{X \geq i\} \leq \sum_{i=n+1}^{\infty} P\{Y \geq i\} \text{ , } n = 1,2, \ldots .$$

As $E[X] = E[Y] = M$ this is equivalent to

$$\sum_{i=1}^{n} P\{X \geq i\} \geq \sum_{i=1}^{n} P\{Y \geq i\} \text{ , } n = 1,2, \ldots .$$

When $n = 1$ the above reduces to $P\{X = 0\} \leq P\{Y = 0\}$. This follows since, as $P\{Y = 1\} \leq P\{X = 1\}$, if $P\{Y = 0\} < P\{X = 0\}$ then it would not be possible for $E[Y]$ to equal $E[X]$. When $n > 1$ the above is equivalent to

$$M \geq \sum_{i=1}^{n} P\{Y \geq i\}$$

which follows since $E[Y] = M$.||

3. A SHOCK MODEL APPLICATION

Suppose that shocks occur in accordance with a renewal process having interarrival distribution G and mean μ_G . Each shock gives rise to a nonnegative random damage which, independent of all else, has probability distribution F . The damages are assumed to be additive and we let $D(t)$ denote the damage at time t . That is

$$D(t) = \sum_{i=1}^{N(t)} X_i$$

where X_i is the damage of the i^{th} shock and $N(t)$ is the number of shocks by t . The system is assumed to fail the first time that $D(t)$ exceeds some constant c . That is, the system fails at time $T_{F,G}$ where

$$T_{F,G} = \min \{t : D(t) > c\} .$$

We will obtain a variability result about $T_{F,G}$ when both F and G are NBUE distributions, where a distribution of a nonnegative random variable X is said to be NBUE (new better than used in expectation) if

$$E[X - t \mid X \geq t] \leq E[X] \quad \text{for all} \quad t \geq 0 .$$

Letting

$$N(c) = \max \{n : X_1 + \ldots + X_n \leq c\}$$

then the system will fail at the time of the $N(c) + 1$ shock.

<u>Lemma 5</u>:

If F is NBUE then

$$N(c) + 1 \underset{v}{\leq} N^*(c) + 1$$

where $N^*(c)$ is a Poisson random variable with mean $c\mu_F$ where $u_F = E[X]$.

Proof:

As $N(c)$ is just the number of renewals by time c of a renewal process whose interarrival distribution is NBUE with mean μ the result follows from Theorem 3.17 on page 173 of [1].||

Proposition 6:

If F and G are both NBUE distributions then

$$T_{F,G} \underset{v}{\leq} T_{E_1,E_2}$$

where E_1 and E_2 are exponential random variables having the same means as F and G respectively.

Proof:

We can express $T_{F,G}$ by

$$T_{F,G} = \sum_{i=1}^{N(c)+1} Y_i$$

where the Y_i , $i \geq 1$, are the interarrival times between successive shocks. They are thus independent and have distribution G . Now it is well known that an NBUE distribution G is less variable than an exponential distribution with the same mean and so

$$Y_i \underset{v}{\leq} \varepsilon_i \text{ when } \varepsilon_i \text{ is exponential with mean } \mu_G .$$

The result now follows from Lemma 5 and Theorem 1.

Remark:

As $T_{E_1,E_2} = \sum_{i=1}^{N^*(c)+1} \varepsilon_i$, it follows upon conditioning on $N^*(c)$ that

$$P\left\{T_{E_1,E_2} \le x\right\} = \sum_{i=0}^{\infty} e^{-\mu_F c} \frac{(\mu_F c)^i}{i!} G_{i+1}(x)$$

where $C_n(x)$ is the gamma distribution with parameters n and $1/\mu_G$ (its mean is $n\mu_G$) . Also if F and G are NBUE then from Proposition 6 all of the moments of $T_{F,G}$ are no greater than the corresponding moments of T_{E_1,E_2} . For instance

$$E[T_{F,G}] \le E\left[T_{E_1,E_2}\right] = E[(N^*(c) + 1)\mu_G] = (c\mu_F + 1)\mu_G .$$

REFERENCES

[1] Barlow, R. and F. Proschan, STATISTICAL THEORY OF RELIABILITY AND LIFE TESTING, Holt, Rinehart and Winston, (1975).

[2] Freedman, D. and R. Purves, "Timid Play is Optimal II," Annals of Mathematical Statistics, Vol. 38, pp. 1284-1285, (1967).

[3] Goodman, L. A., "How to Minimize or Maximize the Probabilities of Extinction in a Galton-Watson Process and in Some Related Multiplicative Population Processes," Annals of Mathematical Statistics, Vol. 39, pp. 1700-1710, (1968).